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SEMI-ALGEBRAIC NEIGHBORHOODS OF CLOSED SEMI-ALGEBRAIC SETS

NICOLAS DUTERTRE

ABSTRACT. Given a closed (not necessarily compact) semi-algebraic set X in \mathbb{R}^n , we construct a nonnegative semi-algebraic C^2 function f such that $X = f^{-1}(0)$ and such that for $\delta > 0$ sufficiently small, the inclusion $X \subset f^{-1}([0, \delta])$ is a retraction. As a corollary, we obtain several formulas for $\chi(X)$.

1. INTRODUCTION

Let X be a compact algebraic set in \mathbb{R}^n . The set X is the set of zeros of a nonnegative polynomial function f . This function f may not be proper as it is explained by the following example due to H. King : let $f(x, y) = (x^2 + y^2)((y(x^2 + 1) - 1)^2 + y^2)$, then $f^{-1}(0) = \{0\}$ but $f(x, (1 + x^2)^{-1}) \rightarrow 0$ as $\|x\| \rightarrow +\infty$.

In [Dur], Durfee proved that any compact algebraic set X can be written as the set of zeros of a proper nonnegative polynomial function g . Following's Thom terminology, he called such a function a rug function for X . Then he defined the notion of algebraic neighborhoods : a subset T with $X \subset T \subset \mathbb{R}^n$ is an algebraic neighborhood of X in \mathbb{R}^n if $T = g^{-1}([0, \delta])$, where g is a rug function for X and δ is a positive real smaller than all nonzero critical values of g . Using the gradient vector field of g , he showed that the inclusion $X \subset T$ is a homotopy equivalence. Thanks to Lojasiewicz's work [Lo1, Lo2] on the trajectories of a gradient vector field, it is not difficult to see that this homotopy equivalence is actually a retraction. Durfee also proved that two algebraic neighborhoods of a compact algebraic set are isotopic. Here also, this uniqueness result is obtained integrating appropriate gradient vector fields.

If X is a non-compact algebraic set in \mathbb{R}^n and f is a nonnegative polynomial such that $X = f^{-1}(0)$, then X is not in general a deformation retract of $f^{-1}([0, \delta])$, where δ is a small regular value of f . Let $f(x, y) = [y(xy - 1)]^2$ (f is the square of the Broughton polynomial [Br] and let $X = f^{-1}(0)$). For δ a sufficiently small positive regular value of f , $f^{-1}([0, \delta])$ has one connected components whereas X has three.

Our aim is to extend Durfee's results to the case of closed (not necessarily compact) semi-algebraic sets. More precisely, we consider a closed semi-algebraic set X in \mathbb{R}^n and an open semi-algebraic neighborhood U of X

in \mathbb{R}^n . We say that $f : U \rightarrow \mathbb{R}$ is an approaching function for X in U (Definition 2.2) if :

- (1) f is semi-algebraic, C^2 , nonnegative,
- (2) $X = f^{-1}(0)$,
- (3) there exists $\delta > 0$ such that $f^{-1}([0, \delta])$ is closed in \overline{U} .

However, the notion of approaching function is not enough to get a deformation retract as it is suggested by the Broughton example above. Let $\rho : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be a proper C^2 semi-algebraic function, let $f : U \rightarrow \mathbb{R}$ be a C^2 nonnegative semi-algebraic function such that $X = f^{-1}(0)$ and let $\Gamma_{f,g}$ be the set of points x in $U \setminus X$ where $\nabla f(x)$ and $\nabla \rho(x)$ are colinear (here ∇f denotes the gradient vector field of f). We say that f is ρ -quasiregular (Definition 2.5) if there does not exist any sequence $(x_k)_{k \in \mathbb{N}}$ of points in $\Gamma_{f,\rho}$ such that $\|x_k\| \rightarrow +\infty$ and $f(x_k) \rightarrow 0$. A ρ -quasiregular approaching semi-algebraic neighborhood of X in U (Definition 3.1) is defined as a set $T = f^{-1}([0, \delta])$ such that :

- (1) f is a ρ -quasiregular approaching function for X in U ,
- (2) δ is a positive number smaller than all nonzero critical values of f ,
- (3) $f^{-1}([0, \delta])$ is closed in \overline{U} ,
- (4) $\Gamma_{f,\rho}$ does not intersect $f^{-1}([0, \delta])$ outside a compact subset K of \mathbb{R}^n .

We prove that ρ -quasiregular approaching semi-algebraic neighborhoods always exist (Corollary 2.8) and that if $T = f^{-1}([0, \delta])$ is a ρ -quasiregular approaching semi-algebraic neighborhood of X in U then X is a strong deformation retract of T (Theorem 3.2). In order to construct this retraction, we study a vector field w that is equal to the gradient of f inside a compact subset of \mathbb{R}^n and to the orthogonal projection of the gradient of f onto the levels of ρ outside a compact set. Using the Łojasiewicz inequality with parameters due to Fekak [Fe] and the usual Łojasiewicz gradient we establish an inequality of “Łojasiewicz’s type” for the norm of w . The retraction is then achieved “pushing” $T = f^{-1}([0, \delta])$ along the trajectories of w .

After we show that two ρ -quasiregular approaching semi-algebraic neighborhoods of X are isotopic (Theorem 4.4). As above, the isotopy is obtained integrating a vector field which is equal to a gradient vector field on a compact set of \mathbb{R}^n and to the projection of this gradient vector field onto the levels of ρ at infinity.

As a corollary, this enables us to prove that when X is smooth of class C^3 , every approaching semi-algebraic neighborhood of X is isotopic to a tubular neighborhood of X (Theorem 5.7).

We end the paper with degree formulas for the Euler-Poincaré characteristic of any closed semi-algebraic set obtained thanks to the machinery developed before (Theorem 6.3, Corollary 6.4 and Corollary 6.5), and with a Petrovskii-Oleinik inequality for the Euler-Poincaré characteristic of any real algebraic set (Proposition 6.8).

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2. ρ -QUASIREGULAR APPROACHING FUNCTIONS

In this section, we define the notion of a ρ -quasiregular approaching function for a closed semi-algebraic set, which generalizes the notion of a rug function introduced by Durfee [Dur].

Let us consider a closed semi-algebraic set X in \mathbb{R}^n . Let U be an open semi-algebraic neighborhood of X . We know that X is the zero set in U of a continuous nonnegative semi-algebraic function $f : U \rightarrow \mathbb{R}$ (for example one can take for f the restriction to U of the distance function to X). For any $\delta > 0$, the set $f^{-1}([0, \delta])$ is closed in U for the induced topology. However, even if δ is very small, it is not necessarily closed in \overline{U} , as it is shown in the following examples.

Example 1. The set $X = \{0\}$ is a closed semi-algebraic set in \mathbb{R} , the set $U =]-1, +\infty[$ is an open semi-algebraic neighborhood of X in \mathbb{R} . Let $f : U \rightarrow \mathbb{R}$ be defined by $f(x) = x^2(x+1)$. It is clear that for any $\delta > 0$, the set $f^{-1}([0, \delta])$ is not closed in $\overline{U} = [-1, +\infty[$.

Example 2. The set $X = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$ is a closed semi-algebraic set in \mathbb{R}^2 , the set $U = \{(x, y) \in \mathbb{R}^2 \mid x^2 y^2 < 1\}$ is an open semi-algebraic neighborhood of X in \mathbb{R}^2 . Let $f : U \rightarrow \mathbb{R}$ be defined by $f(x, y) = y^2$. For any $\delta > 0$, the set $f^{-1}([0, \delta])$ is not closed in $\overline{U} = \{(x, y) \in \mathbb{R}^2 \mid x^2 y^2 \leq 1\}$.

We would like to avoid this situation. For this we need to put a condition on the tuple (X, U, f) .

Definition 2.1. Let X be a closed semi-algebraic set in \mathbb{R}^n , let U be an open neighborhood of X and let $f : U \rightarrow \mathbb{R}$ be a nonnegative continuous semi-algebraic function such that $X = f^{-1}(0)$. We say that (X, U, f) satisfies the condition (A) if there does not exist any sequence $(x_k)_{k \in \mathbb{N}}$ of points in U such that $\lim_{k \rightarrow +\infty} f(x_k) = 0$ and such that $\lim_{k \rightarrow +\infty} x_k$ exists and belongs to $\text{Bd}(U) = \overline{U} \setminus U$.

It is clear that this condition is satisfied when $U = \mathbb{R}^n$. Let us remark that for any couple (X, U) , X being a closed semi-algebraic set in \mathbb{R}^n and U an open semi-algebraic neighborhood of X , there exists a function f such that (X, U, f) satisfies the condition (A). Let $d_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ be the distance function to X and let $d_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ be the distance function to $\text{Bd}(U)$. Let $d : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by

$$d(x) = \frac{d_1(x)}{d_1(x) + d_2(x)}.$$

The tuple $(X, U, d|_U)$ satisfies the condition (A).

We will explain how to construct from a function f such that (X, U, f) satisfies the condition (A), a nonnegative continuous semi-algebraic function g such that $X = g^{-1}(0)$ and $g^{-1}([0, \delta])$ is closed in \overline{U} for δ small enough. Actually we will prove a stronger result.

Let us fix a proper C^2 semi-algebraic function $\rho : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$. We will denote by Σ_r the set $\rho^{-1}(r)$, by D_r the set $\rho^{-1}([0, r])$ and by E_r the set

$\rho^{-1}([r, +\infty[)$. Note that for r sufficiently big, Σ_r is a non-empty compact C^2 -submanifold of \mathbb{R}^n . We will call such a ρ a control function.

Lemma 2.2. *Let X be a closed semi-algebraic set in \mathbb{R}^n , let U be an open semi-algebraic neighborhood of X and let $f : U \rightarrow \mathbb{R}$ be continuous nonnegative semi-algebraic function such that $X = f^{-1}(0)$ and (X, U, f) satisfies the condition (A). For every integer $q \geq 0$, let $f_q : U \rightarrow \mathbb{R}$ be defined by $f_q(x) = (1 + \rho(x))^q \cdot f(x)$. Let $V \subset U$ be an open semi-algebraic neighborhood of X . There exists an integer q_0 such that for every integer $q \geq q_0$, there exists $\delta_q > 0$ such that $f_q^{-1}([0, \delta_q])$ is included in V and closed in \overline{V} . Furthermore, if X is compact then one can choose q_0 such that for every integer $q \geq q_0$, $f_q^{-1}([0, \delta_q])$ is compact in \overline{V} .*

Proof. Let Z be the following closed semi-algebraic set : $Z = \overline{U} \setminus V$. Let $d : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous nonnegative semi-algebraic function such that $X = d^{-1}(0)$ and $Z = d^{-1}(1)$. Let U_1 be the open semi-algebraic neighborhood of X in \mathbb{R}^n defined by $U_1 = d^{-1}([0, \frac{1}{2}])$ and let V_1 be the open semi-algebraic neighborhood of X in U defined by $V_1 = U_1 \cap U$. It is straightforward to see that $\overline{V_1} \subset V$.

Let us study first the case when U is bounded. There exists $\delta > 0$ such that $f^{-1}([0, \delta]) \subset V_1$. Otherwise, we would be able to construct a sequence of points $(x_k)_{k \in \mathbb{N}}$ in $\overline{U} \setminus V_1$ such that $\lim_{k \rightarrow +\infty} f(x_k) = 0$. By compactness of $\overline{U} \setminus V_1$, there would exist a subsequence of points $(x_{\varphi(k)})_{k \in \mathbb{N}}$ in $\overline{U} \setminus V_1$ such that $f(x_{\varphi(k)})$ tends to 0 and $x_{\varphi(k)}$ tends to a point y in $\overline{U} \setminus V_1$. If y belongs to U then $f(y) = 0$, which is impossible. So y belongs to $\overline{U} \setminus U$, which is also impossible by the condition (A). Since $\overline{V_1}$ is included in V and $\overline{V_1}$ is bounded, the set $f^{-1}([0, \delta])$ is compact in \overline{V} .

If U is not bounded and X is not compact, then the following semi-algebraic set $F = U \setminus V_1$ is unbounded as well. There exists r_0 such that for every $r \geq r_0$, $\Sigma_r \cap F$ is not empty (the set $\{r \in \mathbb{R} \mid \Sigma_r \cap F \neq \emptyset\}$ is an unbounded semi-algebraic set of \mathbb{R}). Let $\alpha : [r_0, +\infty[\rightarrow \mathbb{R}$ be defined by

$$\alpha(r) = \inf\{f(x) \mid x \in \Sigma_r \cap F\}.$$

The function α is a semi-algebraic function. Let us show that it is positive. If $\alpha(r) = 0$ then there exists a sequence of points $(x_k)_{k \in \mathbb{N}}$ in $F \cap \Sigma_r$ such that $f(x_k)$ tends to 0. By compactness of Σ_r , we can extract a subsequence $(x_{\varphi(k)})_{k \in \mathbb{N}}$ such that $f(x_{\varphi(k)})$ tends to 0 and $x_{\varphi(k)}$ tends to a point y in $\Sigma_r \cap \overline{F}$, which is included in $\Sigma_r \cap \overline{U}$. If y belongs to U then $f(y) = 0$ and so y belongs to X , which is impossible for $d(y) \geq \frac{1}{2}$. Hence y is in $\text{Bd}(U)$. This is impossible by the condition (A). The function α^{-1} is semi-algebraic. From Proposition 2.11 in [Co1] (see also Proposition 2.6.1 in [BCR]), there exists $r_1 \geq r_0$ and an integer q_0 such that $\alpha(r)^{-1} < r^q$ for every $r \geq r_1$ and every integer $q \geq q_0$. This implies that for every x in $F \cap E_{r_1}$ and for $q \geq q_0$, $f_q(x) = (1 + \rho(x))^q \cdot f(x) > 1$. It is clear that (X, U, f_q) satisfies the condition (A). The same argument as in the case U bounded shows

that there exists ϵ_q such that $f_q^{-1}([0, \epsilon_q]) \cap D_{r_1}$ is included in $V_1 \cap D_{r_1}$. We take for δ_q the minimum of 1 and ϵ_q . Since $\overline{V_1} \subset V$, it is easy to see that $f_q^{-1}([0, \delta_q])$ is closed in \overline{V} .

It remains to study the case U unbounded but X compact. There exists $r_2 > 0$ such that $X \cap E_{r_2}$ is empty. Let $\beta : [r_2, +\infty[\rightarrow \mathbb{R}$ be defined by

$$\beta(r) = \inf\{f(x) \mid x \in U \cap \Sigma_r\}.$$

Thanks to condition (A), we can prove that it is a positive semi-algebraic function. There exists $r_3 \geq r_2$ and an integer q_1 such that $\beta(r)^{-1} < r^q$ for every $r \geq r_3$ and every integer $q \geq q_1$. Hence for $x \in U \cap E_{r_3}$ and for $q \geq q_1$, $f_q(x) = (1 + \rho(x))^q \cdot f(x) > 1$. The tuple (X, U, f_q) satisfies the condition (A). As in the previous cases, there exists $\epsilon_q > 0$ such that $f_q^{-1}([0, \epsilon_q]) \cap D_{r_3}$ is included in $V_1 \cap D_{r_3}$. We take for δ_q the minimum of 1 and ϵ_q . The set $f_q^{-1}([0, \delta_q])$ is compact in $\overline{V_1}$ because it is compact in \mathbb{R}^n . \square

Definition 2.3. *Let X be a closed semi-algebraic set in \mathbb{R}^n and let U be an open semi-algebraic neighborhood of X in \mathbb{R}^n . A function $f : U \rightarrow \mathbb{R}$ is called an approaching function for X in U if :*

- (1) f is semi-algebraic, C^2 , nonnegative,
- (2) $X = f^{-1}(0)$,
- (3) *there exists $\delta > 0$ such that $f^{-1}([0, \delta])$ is closed in \overline{U} . Furthermore if X is compact then $f^{-1}([0, \delta])$ is compact in \overline{U} .*

Proposition 2.4. *Let X be a closed semi-algebraic set in \mathbb{R}^n and let U be an open semi-algebraic neighborhood of X in \mathbb{R}^n . There exist approaching functions for X in U .*

Proof. From [DM] Corollary C.12, it is possible to find a C^2 semi-algebraic function $\phi : \mathbb{R}^n \rightarrow [0, 1]$ such that $X = \phi^{-1}(0)$ and $\text{Bd}(U) = \phi^{-1}(1)$. Let f be the restriction of ϕ to U . The tuple (X, U, f) satisfies the condition (A). Applying Lemma 2.2 to f and U , we can construct approaching functions for X in U . \square

In his study of regularity at infinity of polynomial functions, Tibar introduced the notion of ρ -regularity [Ti]. Let us recall this definition in the real setting.

Definition 2.5. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial function. We say that the fibre $f^{-1}(t_0)$ is ρ -regular at infinity if for any sequence $(x_k) \subset \mathbb{R}^n$ such that $\|x_k\| \rightarrow +\infty$ and $f(x_k) \rightarrow t_0$, there exists k_0 such that if $k \geq k_0$, then $f^{-1}(f(x_k))$ is transverse to $\rho^{-1}(\rho(x_k))$ at x_k .*

Tibar proved that if the fibre $f^{-1}(t_0)$ is ρ -regular at infinity then f is topologically trivial at infinity at t_0 . As a consequence, if an interval $[a, b]$ does not contain any critical value of f and is such that every fibre $f^{-1}(t)$, $t \in [a, b]$, is ρ -regular at infinity, then f is topologically trivial over $[a, b]$.

We will need a slight modification of this definition. For every open semi-algebraic set U and for every C^2 semi-algebraic function $g : U \rightarrow \mathbb{R}$, let $\Gamma_{g, \rho}$

be the semi-algebraic set defined by :

$$\Gamma_{g,\rho} = \{x \in U \mid \nabla g(x) \text{ and } \nabla \rho(x) \text{ are colinear and } g(x) \neq 0\}.$$

Definition 2.6. Let $g : U \rightarrow \mathbb{R}$ be a C^2 semi-algebraic function. We say that g is ρ -quasiregular if there does not exist any sequence $(x_k)_{k \in \mathbb{N}}$ in $\Gamma_{f,\rho}$ such that $\|x_k\|$ tends to infinity and $|g(x_k)|$ tends to 0.

Note that our definition does not imply that $g^{-1}(0)$ has only isolated singularities, unlike Tibar's definition.

Proposition 2.7. Let X be a closed semi-algebraic set in \mathbb{R}^n and let U be an open semi-algebraic neighborhood of X . Let $f : U \rightarrow \mathbb{R}$ be a C^2 semi-algebraic nonnegative function such that $X = f^{-1}(0)$. For every integer q , let $f_q : U \rightarrow \mathbb{R}$ be defined by $f_q = (1 + \rho(x))^q \cdot f(x)$. There exists an integer q_0 such that for every integer $q \geq q_0$, the function f_q is ρ -quasiregular.

Proof. Let r_0 be the greatest critical value of ρ and let $\beta :]r_0, +\infty[\rightarrow \mathbb{R}$ be defined by

$$\beta(r) = \inf\{f(x) \mid x \in \Sigma_r \cap \Gamma_{f,\rho}\}.$$

The function β is semi-algebraic. It is positive since for $r > r_0$, the function $f|_{\Sigma_r \cap U}$ admits a finite number of critical values. As in Lemma 2.2, this implies that there exists $r_1 > r_0$ and an integer q_0 such that for $x \in \Gamma_{f,\rho} \cap E_{r_1}$ and for $q \geq q_0$, $(1 + \rho(x))^q \cdot f(x) > 1$. Since $\Gamma_{f,\rho} = \Gamma_{f_q,\rho}$, every function f_q is ρ -quasiregular for $q \geq q_0$. \square

Corollary 2.8. Let X be a closed semi-algebraic set in \mathbb{R}^n and let U be an open semi-algebraic neighborhood of X . Let $f : U \rightarrow \mathbb{R}$ be a C^2 semi-algebraic nonnegative function such that $X = f^{-1}(0)$. Assume that (X, U, f) satisfies the condition (A). For every integer $q \geq 0$, let $f_q : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by $f_q(x) = (1 + \rho(x))^q \cdot f(x)$. There exists an integer q_0 such that for every $q \geq q_0$, the function is a ρ -quasiregular approaching function for X in U . \square

If X is an algebraic set, it is the zero set of a nonnegative polynomial f . Choosing for ρ a proper nonnegative polynomial and applying the above process, we obtain ρ -quasiregular approaching functions for X that are nonnegative polynomials.

Let us compare our notion of ρ -quasiregular approaching function with the notion of rug function due to Durfee [Dur]. If X is a compact algebraic set of \mathbb{R}^n , a rug function for X is a proper nonnegative polynomial f such that $X = f^{-1}(0)$. It is clear that such a function is a ρ -quasiregular approaching function for X in \mathbb{R}^n .

3. RETRACTION ON A CLOSED SEMI-ALGEBRAIC SET

In this section, we prove that any closed semi-algebraic set is a strong deformation retract of certain closed semi-algebraic neighborhoods of it. First let us specify the closed semi-algebraic neighborhoods that we will consider.

Definition 3.1. Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set, let ρ be a control function and let U be an open semi-algebraic neighborhood of X . A subset T with $X \subset T \subset U$ is a ρ -quasiregular approaching semi-algebraic neighborhood of X in U if $T = f^{-1}([0, \delta])$ where :

- (1) f is a ρ -quasiregular approaching function for X in U ,
- (2) δ is a positive number smaller than all nonzero critical values of f ,
- (3) $f^{-1}([0, \delta])$ is closed in \bar{U} and compact in \bar{U} if X is compact,
- (4) if $\Gamma_{f,\rho}$ is the following polar set :

$$\Gamma_{f,\rho} = \{x \in U \setminus X \mid \nabla f(x) \text{ and } \nabla \rho(x) \text{ are colinear}\},$$

then $\Gamma_{f,\rho}$ does not intersect $f^{-1}([0, \delta])$ outside a compact subset K of \mathbb{R}^n .

For short, we will say that such a T is an approaching semi-algebraic neighborhood. By the results of the previous section, it is clear that approaching semi-algebraic neighborhoods always exist.

Theorem 3.2. Let X be a closed semi-algebraic set and let T be an approaching semi-algebraic neighborhood of X . Then X is a strong deformation retract of T .

Proof. If X is compact, this is already proved by Durfee [Du] and Lojasiewicz [Lo1,Lo2]. So let us assume that X is not compact.

Let us fix f , U , δ , ρ and K which satisfy the conditions of the above definition and such that $T = f^{-1}([0, \delta])$. Furthermore let us assume that $\delta < 1$. We will focus first on the behaviour of f at infinity.

Let $r_0 > 0$ be such that $K \cap E_{r_0}$ is empty and such that Σ_r is a C^2 submanifold for $r \geq r_0$. Let $A = T \cap E_{r_0}$. The set A is a closed semi-algebraic set of \mathbb{R}^n and $A \cap \Gamma_{f,\rho}$ is empty. Let us consider the following closed semi-algebraic set Y of \mathbb{R}^{n+1} :

$$Y = \{(x, t) \in \mathbb{R}^{n+1} \mid x \in A \text{ and } \rho(x) = t\}.$$

We will denote by Y_t the fibre $\{x \in A \mid (x, t) \in Y\}$. Observe that $Y_t = A \cap \Sigma_t$. Let $F : A \rightarrow \mathbb{R}$ be defined by :

$$F(x) = \left\| \nabla f(x) - \left\langle \nabla f(x), \frac{\nabla \rho(x)}{\|\nabla \rho(x)\|} \right\rangle \cdot \frac{\nabla \rho(x)}{\|\nabla \rho(x)\|} \right\|.$$

The function F is just the norm of the orthogonal projection of $\nabla f(x)$ on the manifold $\Sigma_{\rho(x)}$. Moreover it is a continuous semi-algebraic function on A .

Let \tilde{f} and \tilde{F} be the semi-algebraic functions defined on Y by $\tilde{f}(x, t) = f(x)$ and $\tilde{F}(x, t) = F(x)$. They are continuous in x and verify $\tilde{F}^{-1}(0) \subset \tilde{f}^{-1}(0)$. This inclusion is easy to check since $F(x) = 0$ if and only if $\nabla f(x)$ and $\nabla \rho(x)$ are colinear. On A , this can occur only if x belongs to X .

We can apply Lojasiewicz's inequality with parameters due to Fekak (see [Fe], p128). We need some notations : for every t , \tilde{f}_t and \tilde{F}_t are the functions on Y_t defined by $\tilde{f}_t(x) = \tilde{f}(x, t)$ and $\tilde{F}_t(x) = \tilde{F}(x, t)$; for every $S \subset \mathbb{R}$, Y_S denotes the set $Y \cap (\mathbb{R}^n \times S)$. Fekak's theorem states that there exists a finite

partition into semi-algebraic subsets of $\mathbb{R} = \cup S_i$, continuous semi-algebraic functions $h_i : Y_{|S_i} \rightarrow \mathbb{R}$ and rational numbers p_i/q_i such that :

- i) $|\tilde{f}(x, t)|^{p_i/q_i} \leq h_i(x, t)|\tilde{F}(x, t)|$ on $Y_{|S_i}$ for $t \in S_i$,
- ii) p_i/q_i is the Lojasiewicz exponent of \tilde{f}_t with respect to \tilde{F}_t for $t \in S_i$.

Since $\cup S_i$ is a finite semi-algebraic partition of \mathbb{R} , there exist $t_0 \in \mathbb{R}$ and i_0 such that $S_{i_0} = [t_0, +\infty[$. Then for every $t \geq t_0$, we have:

- i) $|\tilde{f}(x, t)|^{p_i/q_i} \leq h_i(x, t)|\tilde{F}(x, t)|$ for $x \in Y_t$,
- ii) p_{i_0}/q_{i_0} is the Lojasiewicz exponent of \tilde{f}_t with respect to \tilde{F}_t .

We know that $\tilde{f}_t = f_{|Y_t}$ and $\tilde{F}_t = \|\nabla(f_{|Y_t})\|$. By Lojasiewicz's gradient inequality applied to $f_{|Y_t}$, we get $p_{i_0}/q_{i_0} < 1$. Let $\alpha = p_{i_0}/q_{i_0}$ and let $B = T \cap E_{t_0}$. We have proved that there exist $0 \leq \alpha < 1$ and a continuous semi-algebraic function $h : B \times [t_0, +\infty[\rightarrow \mathbb{R}$ such that for every $x \in B$:

$$|f(x)|^\alpha \leq h(x, \rho(x)).F(x),$$

where $F(x)$ is the norm of the vector field

$$v(x) = \nabla f(x) - \langle \nabla f(x), \frac{\nabla \rho(x)}{\|\nabla \rho(x)\|} \rangle \cdot \frac{\nabla \rho(x)}{\|\nabla \rho(x)\|}.$$

Let C be the compact semi-algebraic set defined by $C = T \cap D_{2t_0}$. By the Lojasiewicz gradient inequality, there exists $d > 0$ and $0 \leq \beta < 1$ such that on C :

$$|f(x)|^\beta \leq d.\|\nabla f(x)\|.$$

Here we have applied the Kurdyka-Parusinski version of the Lojasiewicz gradient inequality [KP].

We will glue the two vector fields v and ∇f . Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^∞ -function such that :

- $\varphi(x) = 1$ if $\rho(x) \leq 1, 3t_0$,
- $\varphi(x) = 0$ if $\rho(x) \geq 1, 7t_0$,
- $0 < \varphi(x) < 1$ if $1, 3t_0 < \rho(x) < 1, 7t_0$.

Let w be the following vector field on T :

$$w(x) = (1 - \varphi(x))v(x) + \varphi(x)\nabla f(x).$$

We want to find an inequality of "Lojasiewicz's type" for $\|w\|$. First observe that $\|w(x)\| \geq \|v(x)\|$, for

$$w(x) = v(x) + \varphi(x) \cdot \langle \nabla f(x), \frac{\nabla \rho(x)}{\|\nabla \rho(x)\|} \rangle \cdot \frac{\nabla \rho(x)}{\|\nabla \rho(x)\|}.$$

Let M be defined by :

$$M = \max\{h(x, \rho(x)) \mid x \in T \text{ and } 1, 2t_0 \leq \rho(x) \leq 1, 8t_0\}.$$

We have $|f(x)|^\alpha \leq M.\|w(x)\|$ for $x \in T \cap \{x \mid 1, 2t_0 \leq \rho(x) \leq 1, 8t_0\}$. For $x \in T \cap D_{1, 3t_0}$, $|f(x)|^\beta \leq d.\|\nabla f(x)\|$ and $\nabla f(x) = w(x)$. Calling γ the

maximum of α and β and N the maximum of M and d and since $\delta < 1$, we get that for $x \in T \cap D_{1,8t_0}$:

$$|f(x)|^\gamma \leq N \cdot \|w(x)\|. \quad (1)$$

Now for $x \in T \cap E_{1,7t_0}$, $w(x) = v(x)$ and then :

$$|f(x)|^\gamma \leq h(x, \rho(x)) \cdot \|w(x)\|. \quad (2)$$

On one hand, we have :

$$\langle \nabla f(x), w(x) \rangle = (1 - \varphi(x)) \langle \nabla f(x), v(x) \rangle + \varphi(x) \langle \nabla f(x), \nabla f(x) \rangle,$$

hence

$$\langle \nabla f(x), w(x) \rangle = (1 - \varphi(x)) \langle v(x), v(x) \rangle + \varphi(x) \langle \nabla f(x), \nabla f(x) \rangle,$$

since $\langle v(x), \nabla f(x) \rangle = \langle v(x), v(x) \rangle$. On the other hand,

$$\langle w(x), w(x) \rangle = (1 - \varphi(x)^2) \langle v(x), v(x) \rangle + \varphi(x)^2 \langle \nabla f(x), \nabla f(x) \rangle.$$

Using the fact that $0 \leq \varphi(x) \leq 1$, it is easy to see that

$$\langle \nabla f(x), w(x) \rangle \geq \langle w(x), w(x) \rangle \Leftrightarrow \langle \nabla f(x), \nabla f(x) \rangle \geq \langle v(x), v(x) \rangle.$$

Since the inequality on the right hand side is verified, we have proved :

$$\langle \nabla f(x), w(x) \rangle \geq \langle w(x), w(x) \rangle. \quad (3)$$

We are going to integrate the vector field $-\frac{w}{\|w\|}$. Let ϕ_t be the flow associated with the differential equation :

$$\dot{x} = -\frac{w}{\|w\|}.$$

For every $x \in T$, let $b(x) = \sup\{t \mid f(\phi_t(x)) \geq 0\}$ and $\omega(x) = \lim_{t \rightarrow b(x)} \phi_t(x)$. We write $\phi_x(t)$ the trajectory that passes through x . The following facts are proved using inequalities (1), (2) and (3) and adapting to our situation the techniques of Lojasiewicz (see [Lo1], [Lo2], [Ku], [KMP] or [NS] for details).

Fact 1 For all $x \in T$, $\{\phi_x(t) \mid 0 \leq t \leq b(x)\} \subset T$.

Fact 2 For all $x \in T \cap E_{1,7t_0}$, for all t such that $0 \leq t \leq b(x)$, $\|\phi_x(t)\| = \|x\|$.

Fact 3 For all $x \in T \cap D_{1,8t_0}$, for all t such that $0 \leq t \leq b(x)$, $\|\phi_x(t)\| \leq 1, 8t_0$.

Fact 4 For all $x \in T$, $b(x) < +\infty$.

Fact 5 For all $x \in T$, $f(\omega(x)) = 0$.

Fact 6 The mapping $\omega : T \rightarrow X$, $x \mapsto \omega(x)$ is continuous.

Fact 7 The mapping $b : T \rightarrow X$, $x \mapsto b(x)$ is continuous.

Now we can end the proof of Theorem 3.2. The retraction is given by the following mapping :

$$\begin{aligned} G : [0, 1] \times T &\rightarrow T \\ (t, x) &\mapsto G(t, x) = \phi(tb(x), x). \end{aligned}$$

If $\delta \geq 1$, we can push $f^{-1}([0, \delta])$ onto $f^{-1}([0, \delta'])$, $\delta' < 1$, along the trajectories of V . \square

We end this section with a remark. Using the same method, one can prove the following result. Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a nonnegative semi-algebraic function such that $X = f^{-1}(0)$. Let $\Gamma_{f,\rho}$ be the set defined by

$$\Gamma_{f,\rho} = \{x \in \mathbb{R}^n \mid \nabla f(x) \text{ and } \nabla \rho(x) \text{ are not colinear and } f(x) \neq 0\}.$$

Let r be a regular value of ρ . Assume that the following assumption is satisfied : there is no sequence of points (x_k) in $\Gamma_{f,\rho} \cap D_r$ such that $\rho(x_k) \rightarrow r$ and $f(x_k) \rightarrow 0$. Then for $\delta > 0$ sufficiently small, the inclusion $X \cap D_r \subset f^{-1}([0, \delta]) \cap D_r$ is a deformation retract.

For example, this result can be applied if f has only isolated critical points on its zero level and X intersects Σ_r transversally.

4. UNIQUENESS OF ρ -QUASIREGULAR APPROACHING NEIGHBORHOOD

In this section, we prove that two ρ -quasiregular approaching semi-algebraic neighborhoods of a closed non-compact semi-algebraic set are isotopic. Let us recall what Durfee proved in the case of a compact semi-algebraic set.

Theorem 4.1. *Let X be a compact semi-algebraic set, let f_1 and f_2 be two rug functions for X . Let δ_1 (resp. δ_2) be a positive regular value of f_1 (resp. f_2) smaller than all nonzero critical values of f_1 (resp. f_2). Let $T_1 = f_1^{-1}([0, \delta_1])$ and $T_2 = f_2^{-1}([0, \delta_2])$. There is a continuous family of homeomorphisms $h_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $0 \leq t \leq 1$, such that :*

- (1) h_0 is the identity,
- (2) for all t , $h_t|_X$ is the identity,
- (3) $h_1(T_1) = T_2$ and h_1 is a smooth diffeomorphism of $T_1 \setminus X$ onto $T_2 \setminus X$.

Proof. See [Dur], Proposition 1.7. Actually Durfee considers the case of a compact algebraic set but his proof works in our case. \square

We will prove the following theorem.

Theorem 4.2. *Let X be a closed non-compact semi-algebraic set and let ρ be a control function. If T_1 and T_2 are two ρ -quasiregular approaching semi-algebraic neighborhoods of X in U then there is a continuous family of homeomorphisms $h_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $0 \leq t \leq 1$, such that :*

- (1) h_0 is the identity,
- (2) for all t , $h_t|_X$ is the identity,
- (3) $h_1(T_1) = T_2$ and h_1 is a smooth diffeomorphism of $T_1 \setminus X$ onto $T_2 \setminus X$.

Proof. Let us write $T_i = f_i^{-1}([0, \delta_i])$ where f_i is a ρ -quasiregular approaching function for X in U_i , $i = 1, 2$. We will prove our result adapting the ideas of Durfee [Dur]. There are three steps.

Let us first consider the case $f_1 = f_2 = f$ and $U_1 = U_2 = U$. We can assume without loss of generality that $\delta_1 < \delta_2$. Thanks to condition (4) in Definition 3.1, we see that $f^{-1}(\delta)$ is ρ -regular at infinity for every δ in $[\delta_1, \delta_2]$. Since $[\delta_1, \delta_2]$ does not contain any critical value of f , Tibar's work implies that T_1 and T_2 are diffeomorphic. Let us be more precise and explain how the family h_t is obtained. As we did in the proof of Theorem 3.2, we can construct a vector field w on $f^{-1}([\delta_1, \delta_2])$ which is equal to the orthogonal projection of ∇f on the levels of ρ outside a set D_R , and equal to ∇f inside a set $D_{R'}$, $R' < R$. Then we extend w to a complete vector field \tilde{w} defined on \mathbb{R}^n using a smooth function equal to 1 on the closed set $f^{-1}([\delta_1, \delta_2])$ and to 0 on the closed set $X \cup (\mathbb{R}^n \setminus U)$. Integrating this vector field gives the required family h_t .

The second case is when $f_2 = (1 + \rho)^q f_1$ and $U_1 = U_2 = U$. Let δ be the minimum of δ_1 and δ_2 . Let v_1 (resp. v_2) be the orthogonal projection of ∇f_1 on the levels of ρ . By condition (4) in Definition 3.1, there exists $R > 0$ such that v_1 and v_2 do not vanish in $f_1^{-1}([0, \delta]) \cap E_R$. It is clear that on this set, they do not point in opposite direction. There exists a neighborhood U' of $X \cap D_{2R}$ in D_{2R} such that ∇f_1 and ∇f_2 are nonzero and do not point in opposite direction on $U' \setminus X$. This fact is proved in the same way as Lemma 1.8 in [Dur]. Hence there exists δ' such that ∇f_1 and ∇f_2 are nonzero and do not point in opposite direction on $f_1^{-1}([0, \delta']) \cap D_{2R}$. Let δ'' be the minimum of δ and δ' . By the first case, it is enough to prove that $f_2([0, \delta''])$ and $f_1([0, \delta''])$ are isotopic. Let S be the closed set $f_1^{-1}([0, \delta'']) \setminus f_2^{-1}([0, \delta''])$ and let $g : S \rightarrow [0, 1]$ be defined by :

$$g(x) = \frac{f_2(x) - \delta''}{f_2(x) - f_1(x)}.$$

Note that $g^{-1}(0) = f_2^{-1}(\delta'')$ and $g^{-1}(1) = f_1^{-1}(\delta'')$. The gradient of g is

$$\nabla g(x) = \frac{(f_2(x) - \delta'')\nabla f_1(x) + (\delta'' - f_1(x))\nabla f_2(x)}{(f_2(x) - f_1(x))^2}.$$

Let v be its orthogonal projection on the levels of ρ . It is nonzero in $S \cap E_R$. Moreover, ∇g is nonzero in $S \cap D_{2R}$. Gluing these two vector fields, we obtain a C^1 vector field w on S and we proceed as in the first case.

The third case is the general case. Let $U = U_1 \cap U_2$. By Lemma 2.2 and the second case above, we can assume that $T_1 \subset U$, $T_2 \subset U$ and T_1 and T_2 are closed in \overline{U} . We need some lemmas.

Lemma 4.3. *For every integer $q \geq 0$, let $f_{1q} : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by $f_{1q}(x) = (1 + \rho(x))^q \cdot f_1(x)$. Let v_{1q} (resp. v_2) be the orthogonal projection of ∇f_{1q} (resp. ∇f_2) on the levels of ρ . There exist $q_0 \in \mathbb{N}$ and $R > 0$ such that for all $q \geq q_0$ the vector fields v_{1q} and v_2 are nonzero and do not point in*

opposite direction in $f_{1q}^{-1}([0, \delta_q]) \cap E_R$, where δ_q is a small regular value of f_{1q} such that $f_{1q}^{-1}([0, \delta_q]) \subset U$ and $f_{1q}^{-1}([0, \delta_q])$ is closed in \overline{U} .

Proof. We known that there exists $R' > 0$ and $U' \subset U$ such that v_1 and v_2 do not vanish in $U' \cap E_{R'}$ since f_1 and f_2 are ρ -quasiregular. Let $\Gamma_{f_1, \rho}$, $\Gamma_{f_2, \rho}$ and $\Gamma_{f_1, f_2, \rho}$ be the following semi-algebraic sets :

$$\Gamma_{f_1, \rho} = \{x \in U \setminus X \mid v_1(x) = 0\}, \quad \Gamma_{f_2, \rho} = \{x \in U \setminus X \mid v_2(x) = 0\},$$

$$\Gamma_{f_1, f_2, \rho} = \{x \in U \setminus X \mid v_1(x) \text{ and } v_2(x) \text{ point in opposite direction}\},$$

and let Γ be the union of these three sets. Let r_0 be the greatest critical value of ρ and let $\alpha :]r_0, +\infty[\rightarrow \mathbb{R}$ be defined by $\alpha(r) = \inf \{f_1(x) \mid x \in \Sigma_r \cap \Gamma\}$. Then α is a positive semi-algebraic function. To see that it is positive, it is enough to apply Lemma 1.8 of [Dur] to the semi-algebraic subset $X \cap \Sigma_r$ of the smooth semi-algebraic set Σ_r . As in Lemma 2.2, this implies that there exists $R > r_0$ and an integer q_0 such that for $x \in \Gamma \cap E_R$ and for $q \geq q_0$, $(1 + \rho(x))^q \cdot f_1(x) > 1$. Since $v_{1q} = (1 + \rho)^q \cdot v_1$, we see that $\Gamma_{f_1, f_2, \rho} = \Gamma_{f_{1q}, f_{2q}, \rho}$. We take δ_q to be the minimum of δ_1 and 1. This ends the proof of Lemma 4.3. \square

Lemma 4.4. *For every integer $q \geq 0$, let $f_{1q} : \mathbb{R}^n \rightarrow \mathbb{R}$ (resp. $f_{2q} : \mathbb{R}^n \rightarrow \mathbb{R}$) be defined by $f_{1q}(x) = (1 + \rho(x))^q \cdot f_1(x)$ (resp. $f_{2q}(x) = (1 + \rho(x))^q \cdot f_2(x)$). Let v_{1q} (resp. v_{2q}) be the orthogonal projection of ∇f_{1q} (resp. ∇f_{2q}) on the levels of ρ . There exist $q_0 \in \mathbb{N}$ and $R > 0$ such that for all $q \geq q_0$ and for all $l \in \mathbb{N}$ the vector fields v_{1q} and v_{2l} are nonzero and do not point in opposite direction in $f_{1q}^{-1}([0, \delta_q]) \cap E_R$, where δ_q is a small regular value of f_{1q} such that $f_{1q}^{-1}([0, \delta_q]) \subset U$ and $f_{1q}^{-1}([0, \delta_q])$ is closed in \overline{U} .*

Proof. It is clear because $v_{2l} = (1 + \rho)^l \cdot v_2$ and $\Gamma_{f_{1q}, f_{2l}, \rho} = \Gamma_{f_{1q}, f_2, \rho}$. This ends the proof of Lemma 4.4. \square

Let us fix q and δ_q which satisfy the conclusion of Lemma 4.3. Applying Lemma 2.2 to the open semi-algebraic neighborhood $f_{1q}^{-1}([0, \delta_q])$ of X and the approaching function f_2 , we can find l such that $f_{2l}^{-1}([0, \epsilon_l]) \subset f_{1q}^{-1}([0, \delta_q])$, where ϵ_l is a small regular value of f_{2l} . Thanks to Lemma 4.4, we can proceed as we did for the second case, namely we consider the closed set $S' = f_{1q}^{-1}([0, \delta_q]) \setminus f_{2l}^{-1}([0, \epsilon_l])$ and the function $h : S' \rightarrow [0, 1]$ defined by:

$$h(x) = \frac{f_{2l}(x) - \epsilon_l}{(f_{2l}(x) - \epsilon_l) - (\delta_q - f_{1q}(x))}.$$

This ends the proof of Theorem 4.2. \square

We end this section with this question : if X is a closed non-compact semi-algebraic set in \mathbb{R}^n , if ρ_1 and ρ_2 are two distinct control functions, are a ρ_1 -quasiregular approaching semi-algebraic neighborhood and a ρ_2 -quasiregular approaching semi-algebraic neighborhood of X isotopic ?

5. THE SMOOTH CASE

In this section, we assume that X is a closed non-compact semi-algebraic set in \mathbb{R}^n and also a C^3 submanifold of dimension $k < n$. We also assume that ρ is a control function of class C^3 . We show that any ρ -quasiregular approaching semi-algebraic neighborhood of X is isotopic to a tubular neighborhood of X . For this, we construct a kind of distance function to X which is C^2 in a semi-algebraic neighborhood of X and ρ -quasiregular.

Let us fix X and ρ satisfying the above assumptions. Let $r_0 > 0$ be such that for all $r \geq r_0$, Σ_r is a C^3 submanifold that intersects X transversally. Let F be the following set :

$$F = \{(x, v) \in X \times \mathbb{R}^n \mid \rho(x) > r_0, \langle v, \nabla \rho(x) \rangle = 0 \\ \text{and } \langle v, w \rangle = 0 \text{ for all } w \in T_x(X \cap \Sigma_{\rho(x)})\}.$$

It is a C^2 -vector bundle over $X \cap \{x \mid \rho(x) > r_0\}$ whose fibers are $n - k$ -dimensional. Moreover it is semi-algebraic. We will denote the fiber over x by F_x . Observe that F_x is the normal space of $X \cap \Sigma_{\rho(x)}$ in $\Sigma_{\rho(x)}$.

Let N be the normal bundle over $X \cap \{x \mid \rho(x) < 2r_0\}$:

$$N = \{(x, v) \in X \times \mathbb{R}^n \mid \rho(x) < 2r_0 \text{ and } v \perp T_x X\}.$$

It is also a C^2 semi-algebraic vector bundle. We denote the fiber over x by N_x .

We will glue these two bundles. By [DM] Corollary C.12, it is possible to find a C^2 semi-algebraic function $\phi : X \mapsto [0, 1]$ such that $X \cap E_{7/4r_0} = \phi^{-1}(1)$ and $X \cap D_{5/4r_0} = \phi^{-1}(0)$. For each x such that $r_0 < \rho(x) < 2r_0$, let P_x be restriction to F_x of the orthogonal projection to N_x .

We can define a bundle $H \subset X \times \mathbb{R}^n$ over X in the following way :

$$\begin{aligned} &\text{if } \rho(x) < 5/4r_0 \text{ then } H_x = N_x, \\ &\text{if } r_0 < \rho(x) < 2r_0 \text{ then } H_x = \{v \in \mathbb{R}^n \mid \exists w \in F_x \\ &\quad \text{such that } v = \phi(x)w + (1 - \phi(x))P_x(w)\}, \\ &\text{if } \rho(x) > 7/4r_0 \text{ then } H_x = F_x. \end{aligned}$$

It is an exercise of linear algebra to prove that H is a vector bundle whose fibres are $n - k$ -dimensional planes. Furthermore, it is C^2 semi-algebraic because F and N are C^2 semi-algebraic bundles and because ϕ is a C^2 semi-algebraic function. This bundle H will enable us to construct the desired “distance” function to X . Let φ be the following C^2 semi-algebraic mapping :

$$\begin{aligned} \varphi : H &\rightarrow \mathbb{R}^n \\ (x, v) &\mapsto x + v. \end{aligned}$$

Then there exists a semi-algebraic open neighborhood U of the zero-section $X \times \{0\}$ in H such that the restriction $\varphi|_U$ is a C^2 diffeomorphism onto a neighborhood V of X . Moreover, we can take U of the form :

$$U = \{(x, v) \mid \|v\| < \varepsilon(x)\},$$

where ε is a positive C^2 semi-algebraic function on X . The proof of this result is given in [Co2], Lemma 6.15 for the normal bundle. This proof actually holds in our case. This provides us with a C^2 semi-algebraic retraction $\pi : V \rightarrow X$ and a C^2 semi-algebraic distance function $d' : V \rightarrow X$ defined by $\pi(\varphi(x, v)) = x$ and $d'(\varphi(x, v)) = \|v\|^2$.

Lemma 5.1. *There exists an open semi-algebraic neighborhood W of X in V such that for every $y \in W$, $\rho(y) \leq 1, 1\rho(\pi(y))$. Furthermore, one can choose W of the form :*

$$W = \{y \in V \mid d'(y) < \varepsilon'(\pi(y))\},$$

where $\varepsilon' : X \rightarrow \mathbb{R}$ is a positive C^2 semi-algebraic function.

Proof. Let A be the following semi-algebraic set :

$$A = \{y \in V \mid \rho(y) > 1, 1\rho(\pi(y))\}.$$

Let $\alpha : \pi(A) \rightarrow \mathbb{R}$ be the function defined as follows :

$$\alpha(x) = \inf \{d'(y) \mid y \in \pi^{-1}(x) \cap A\}.$$

This is a semi-algebraic function on $\pi(A)$. Let us prove that it is positive. The continuity of $\rho \circ \varphi$ implies that for every x in $\pi(A)$, there exists δ_x with $0 < \delta_x < \varepsilon(x)$, such that for every v in H_x with $\|v\| \leq \delta_x$, $\rho(\varphi(x, v)) \leq 1, 1\rho(\varphi(x, 0))$. Since $\|v\|^2 = d'(y)$ if $y = \varphi(x, v)$, this proves that $\alpha(x) \geq \delta_x > 0$. Let us show that α is locally bounded from below by positive constants, i.e for every $x \in \pi(A)$, there exist $c > 0$ and a neighborhood Ω of x in $\pi(A)$ such that $\alpha > c$ on Ω . If it is not the case, we can find a sequence of points x_n in $\pi(A)$ tending to x such that $\alpha(x_n)$ tends to 0. Hence there exists a sequence of points $y_n = \varphi(x_n, v_n)$ such that v_n tends to 0, x_n tends to x and $\rho(\varphi(x_n, v_n)) > 1, 1\rho(\varphi(x_n, 0))$. By continuity, we obtain $\rho(\varphi(x, 0)) \geq 1, 1\rho(\varphi(x, 0))$, which is impossible. Let $\tilde{\alpha} : X \rightarrow \mathbb{R}$ be defined by $\tilde{\alpha}(x) = \alpha(x)$ if $x \in \pi(A)$ and $\tilde{\alpha}(x) = \varepsilon(x)$ if $x \notin \pi(A)$. The function $\tilde{\alpha}$ is semi-algebraic, positive and locally bounded from below by positive constants. Applying Lemma 6.12 of [Co2], we can find a positive semi-algebraic C^2 function $\varepsilon' : X \rightarrow \mathbb{R}$ such that $\varepsilon' < \tilde{\alpha}$ on X . \square

Let us study the function $d' : W \rightarrow \mathbb{R}$ more precisely. Let B be the following semi-algebraic set :

$$B = \left\{ y \in W \cap E_{2r_0} \mid \frac{\langle \nabla \rho(y), \nabla \rho(\pi(y)) \rangle}{\|\nabla \rho(y)\| \cdot \|\nabla \rho(\pi(y))\|} < 0, 9 \right\}.$$

Let $\beta : \pi(B) \rightarrow \mathbb{R}$ be the function defined as follows :

$$\beta(x) = \inf \{d'(y) \mid y \in \pi^{-1}(x) \cap B\}.$$

This is a semi-algebraic function on $\pi(B)$ and for every $x \in \pi(B)$, $\beta(x) \leq \varepsilon'(x)$. The same argument as in the above lemma shows that β is positive and locally bounded from below by positive constants. Let $\tilde{\beta} : X \rightarrow \mathbb{R}$ be defined by $\tilde{\beta}(x) = \beta(x)$ if $x \in \pi(B)$ and $\tilde{\beta}(x) = \varepsilon'(x)$ if $x \notin \pi(B)$. The function $\tilde{\beta}$ is semi-algebraic, positive and locally bounded from below

by positive constants. We can find a positive semi-algebraic C^2 function $\varepsilon'' : X \rightarrow \mathbb{R}$ such that $\varepsilon'' < \tilde{\beta}$ on X .

Let W' be defined by :

$$W' = \{y \in V \mid d'(y) < \varepsilon''(\pi(y))\}.$$

Note that W' is included in W . For every y in $W' \cap E_{2r_0}$, we have :

$$\frac{\langle \nabla \rho(y), \nabla \rho(\pi(y)) \rangle}{\|\nabla \rho(y)\| \cdot \|\nabla \rho(\pi(y))\|} \geq 0, 9.$$

Since $\nabla d'(y)$ belongs to $[\nabla \rho(\pi(y))]^\perp$, this can be reformulated in the following way : for every y in $W' \cap E_{2r_0}$, we have :

$$\frac{\langle \nabla \rho(y), \nabla d'(y) \rangle}{\|\nabla \rho(y)\| \cdot \|\nabla d'(y)\|} \leq \sqrt{0, 19}.$$

Lemma 5.2. *There exist $q_0 \in \mathbb{N}$ and $r'_0 > 0$ such that for every $q \geq q_0$ and for every $x \in X \cap E_{r'_0}$,*

$$\frac{1}{(1 + \rho(x))^q} \leq \varepsilon''(x).$$

Proof. Let $h : [0, +\infty[\rightarrow \mathbb{R}$ be defined by :

$$h(r) = \min \{\varepsilon''(x) \mid x \in X \cap \Sigma_r\}.$$

Since h is a positive semi-algebraic function, there exists an integer q_0 and a real $r'_0 > 0$ such that $\frac{1}{h(r)} < r^{q_0}$ for every $r \geq r'_0$. Hence for every $q \geq q_0$ and every $x \in X \cap E_{r'_0}$, we have :

$$\frac{1}{(1 + \rho(x))^q} \leq \varepsilon''(x).$$

□

Corollary 5.3. *There exist $q_0 \in \mathbb{N}$ and $r_0'' > 0$ such that for every $q \geq q_0$ and for every $y \in W' \cap E_{r_0''}$,*

$$\frac{1}{(1 + \rho(\pi(y)))^q} \leq \varepsilon''(\pi(y)).$$

Proof. By Lemma 5.1, we can find $r_0'' > 0$ such that $\pi(y)$ belongs to $X \cap E_{r'_0}$ if y belongs to $W' \cap E_{r_0''}$. □

Lemma 5.4. *There exist $q_1 \in \mathbb{N}$ and $r'_1 > 0$ such that for every $q \geq q_1$ and for every $x \in X \cap E_{r'_1}$, $\|\nabla \rho(x)\| \leq (1 + \rho(x))^q$.*

Proof. Let $c > 0$ be such that $[c, +\infty[$ does not contain any critical value of ρ . Let $l : [c, +\infty[\rightarrow \mathbb{R}$ be defined by :

$$l(r) = \max \{\|\nabla \rho(x)\| \mid x \in X \cap \Sigma_r\}.$$

Since l is a positive semi-algebraic function, there exists an integer q_1 and a real $r'_1 > 0$ such that $l(r) < r^{q_1}$ for every $r \geq r'_1$. Hence for every $q \geq q_1$ and every $x \in X \cap E_{r'_1}$, we have $\|\nabla \rho(x)\| \leq (1 + \rho(x))^q$. □

Corollary 5.5. *There exist $q_1 \in \mathbb{N}$ and $r_1'' > 0$ such that for every $q \geq q_1$ and for every $y \in W' \cap E_{r_1''}$, $\|\nabla \rho(\pi(y))\| \leq (1 + \rho(\pi(y)))^q$.*

Proof. The proof is the same as Corollary 5.3. \square

Proposition 5.6. *There exists an integer q_2 such that for every $q \geq q_2$, the function $d'_q : W' \rightarrow \mathbb{R}$ defined by $d'_q(y) = (1 + \rho(\pi(y)))^q d'(y)$ is a ρ -quasiregular approaching function for X in W' .*

Proof. Since $W' = \{y \in V \mid d'(y) < \varepsilon''(\pi(y))\}$ and ε'' is a positive function, (X, W', d') satisfies the condition (A). Let

$$W_1 = \left\{ y \in V \mid d'(y) < \frac{1}{2} \varepsilon''(\pi(y)) \right\}.$$

We have $\overline{W_1} \subset W'$. By Corollary 5.3, for every $q \geq q_0$, the set $E_{r_0''} \cap d_q^{-1}([0, \frac{1}{4}])$ is included in W_1 . The tuple (X, W', d'_q) satisfies the condition (A). As it has been already explained in Lemma 2.2, there exists $\epsilon_q > 0$ such that $d_q^{-1}([0, \epsilon_q]) \cap D_{r_0''} \subset W_1 \cap D_{r_0''}$. Let δ_q be the minimum of $\frac{1}{4}$ and ϵ_q . The set $d_q^{-1}([0, \delta_q])$ is included in W_1 , hence closed in $\overline{W_1}$ and in $\overline{W'}$. This proves that d'_q is an approaching function for X in W' .

Let us show that it is ρ -quasiregular. Let us fix r greater than r_0'' , r_1'' and $2r_0$ and let us fix q_2 greater than q_0 and q_1 . For every y in $W \cap E_r$, let P_y be the orthogonal projection onto the space $\nabla \rho(y)^\perp$. We have :

$$\nabla d'_q(y) = (1 + \rho(\pi(y)))^{q-1} \cdot [(1 + \rho(\pi(y))) \nabla d'(y) + q d'(y) \nabla \rho(\pi(y))],$$

hence,

$$\frac{P_y(\nabla d'_q(y))}{(1 + \rho(\pi(y)))^{q-1}} = (1 + \rho(\pi(y))) P_y(\nabla d'(y)) + q d'(y) P_y(\nabla \rho(\pi(y))).$$

Let us prove that, for $q \geq q_2$ and $R \geq r$ sufficiently big, $T(y)$ can not vanish if y belongs to $d_q^{-1}([0, 1]) \cap E_R$, where

$$T(y) = (1 + \rho(\pi(y))) P_y(\nabla d'(y)) + q d'(y) P_y(\nabla \rho(\pi(y))).$$

First observe that if y lies in $d_q^{-1}([0, 1]) \cap E_R$, $q \geq q_2$ and $R \geq r$, then :

$$\frac{\langle \nabla \rho(y), \nabla \rho(\pi(y)) \rangle}{\|\nabla \rho(y)\| \cdot \|\nabla \rho(\pi(y))\|} \geq 0,9,$$

and

$$\frac{\langle \nabla \rho(y), \nabla d'(y) \rangle}{\|\nabla \rho(y)\| \cdot \|\nabla d'(y)\|} \leq \sqrt{0,19}.$$

This implies that

$$\|P_y(\nabla \rho(\pi(y)))\| \leq \sqrt{0,19} \|\nabla \rho(\pi(y))\|$$

and

$$\|P_y(\nabla d'(y))\| \geq 0,9 \|\nabla d'(y)\|.$$

Therefore, we have :

$$\|q d'(y) P_y(\nabla \rho(\pi(y)))\| \leq \sqrt{0,19} q d'(y) \|\nabla \rho(\pi(y))\|,$$

and,

$$\| (1 + \rho(\pi(y))) P_y(\nabla d'(y)) \| \geq 0,9 (1 + \rho(\pi(y))) \cdot \|\nabla d'(y)\|,$$

that is to say :

$$\| (1 + \rho(\pi(y))) P_y(\nabla d'(y)) \| \geq 0,9 (1 + \rho(\pi(y))) \cdot 2\sqrt{d'(y)}.$$

In order to prove that $T(y)$ does not vanish if $y \in d_q'^{-1}([0, 1]) \cap E_R$ for $q \geq q_2$ and $R \geq r$ sufficiently big, it is enough to prove that :

$$\frac{1,8}{\sqrt{0,19}} > \frac{q\sqrt{d'(y)}\|\nabla\rho(\pi(y))\|}{1 + \rho(\pi(y))}.$$

But if $y \in d_q'^{-1}([0, 1]) \cap E_R$ where $q \geq q_2$ and $R \geq r$ then we have :

$$\sqrt{d'(y)} \leq \frac{1}{(1 + \rho(\pi(y)))^{\frac{q}{2}}}.$$

So, if we show that

$$\frac{1,8}{\sqrt{0,19}} > \frac{q\|\nabla\rho(\pi(y))\|}{(1 + \rho(\pi(y)))^{\frac{q}{2}+1}},$$

then the required result is established. Let q be such that $\frac{q}{2} + 1 > q_1$. By Corollary 5.5, we have :

$$\frac{q\|\nabla\rho(\pi(y))\|}{(1 + \rho(\pi(y)))^{\frac{q}{2}+1}} \leq \frac{q}{(1 + \rho(\pi(y)))^{\frac{q}{2}+1-q_1}},$$

for $y \in d_q'^{-1}([0, 1]) \cap E_R$, $R \geq r$. Lemma 5.1 implies that there exists $R_q \geq r$ such that if y belongs to $d_q'^{-1}([0, 1]) \cap E_R$, with $R \geq R_q$, then we have :

$$\frac{q}{(1 + \rho(\pi(y)))^{\frac{q}{2}+1-q_1}} < \frac{1,8}{\sqrt{0,19}}.$$

This proves the proposition. \square

We can state the main result of this section.

Theorem 5.7. *Let X be a closed non-compact semi-algebraic set in \mathbb{R}^n which is a C^3 submanifold. Let ρ be a control function of class C^3 . Any ρ -quasiregular approaching semi-algebraic neighborhood of X is isotopic to a tubular neighborhood of X .*

Proof. By the previous proposition, we known that there exist ρ -quasiregular approaching functions d_q' for X in W of the form $d_q'(y) = (1 + \rho(\pi(y)))^q \cdot d'(y)$. But for $\nu > 0$ sufficiently small the set $d_q'^{-1}([0, \nu])$ is a tubular neighborhood of X . It is enough to use Theorem 4.2 to conclude. \square

6. DEGREE FORMULAS FOR THE EULER-POINCARÉ CHARACTERISTIC OF A CLOSED SEMI-ALGEBRAIC SET

In this section, we give degree formulas for the Euler-Poincaré characteristic of a closed semi-algebraic set X included in \mathbb{R}^n . When X is algebraic, we deduce from these formulas a Petrovskii-Oleinik inequality for $|1 - \chi(X)|$.

Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a nonnegative C^2 semi-algebraic function such that $X = f^{-1}(0)$, i.e f is an approaching function for X in \mathbb{R}^n . Let ρ be a control function. For every $q \in \mathbb{N}$, we will denote by f_q the function defined by $f_q(x) = (1 + \rho(x))^q \cdot f(x)$. We will also denote by $\Gamma_{f,\rho}$ (resp. $\Gamma_{f_q,\rho}$) the following polar set :

$$\Gamma_{f,\rho} = \{x \in \mathbb{R}^n \setminus X \mid \nabla f(x) \text{ (resp. } \nabla f_q(x)) \text{ and } \nabla \rho(x) \text{ are colinear}\}.$$

Note that $\Gamma_{f,\rho} = \Gamma_{f_q,\rho}$ for each $q \in \mathbb{N}$. The following proposition is similar to Proposition 2.7 and is proved in the same way.

Proposition 6.1. *There exists an integer q_0 such that for every $q \geq q_0$, the following property holds : for any sequence $(x_k)_{k \in \mathbb{N}} \subset \Gamma_{f_q,\rho}$ such that $\lim_{k \rightarrow +\infty} \|x_k\| = +\infty$, we have $\lim_{k \rightarrow +\infty} f_q(x_k) = +\infty$.*

Let us fix an integer q satisfying the property of the previous proposition. Let $\Sigma(f_q)$ be the set of critical points of f_q and let $\Sigma^*(f_q)$ be the set of critical points of f_q lying in $\mathbb{R}^n \setminus X$.

Corollary 6.2. *The set $\Sigma^*(f_q)$ is compact.*

Proof. It is clearly closed as the set of preimages of the nonzero critical values of f_q . If it is not bounded, there exists a sequence of points $(x_k)_{k \in \mathbb{N}}$ such that $x_k \notin X$, $\nabla f_q(x_k) = 0$ and $\lim_{k \rightarrow +\infty} \|x_k\| = +\infty$. Since for each $k \in \mathbb{N}$, x_k also belongs to $\Gamma_{f_q,\rho}$, this gives a contradiction. \square

Let us decompose $\Sigma^*(f_q)$ into the finite union of its connected components $K_1^q, \dots, K_{m_q}^q$:

$$\Sigma^*(f_q) = \bigcup_{i=1}^{m_q} K_i^q.$$

Before stating the main results of this section, we need to introduce some notations. For each $i \in \{1, \dots, m_q\}$, let U_i be a relatively compact neighborhood of K_i^q such that ∂U_i is a smooth hypersurface and $U_i \cap \Sigma^*(f_q) = K_i^q$. For any mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $F^{-1}(0) \cap U_i = K_i^q$ or $F^{-1}(0) \cap U_i$ is empty, we will denote by $\deg_{K_i^q} F$ the topological degree of the following mapping :

$$\begin{array}{ccc} \frac{F}{\|F\|} & : & \partial U_i \rightarrow S^{n-1} \\ x & \mapsto & \frac{F(x)}{\|F(x)\|}. \end{array}$$

It is well-known that this topological degree does not depend on the choice of the relatively compact neighborhood U_i .

Theorem 6.3. *The Euler-Poincaré characteristic of X is related to ∇f_q by the following formula :*

$$\chi(X) = 1 - \sum_{i=1}^{m_q} \deg_{K_i^q} \nabla f_q.$$

Proof. By Proposition 6.1, f_q is a ρ -quasiregular approaching function for X in \mathbb{R}^n . Theorem 3.2 implies that for $\varepsilon > 0$ sufficiently small :

$$\chi(X) = \chi(\{f_q \leq \varepsilon\}).$$

By the Mayer-Vietoris sequence, we have :

$$1 = \chi(\{f_q \leq \varepsilon\}) + \chi(\{f_q \geq \varepsilon\}) - \chi(\{f_q = \varepsilon\}). \quad (1)$$

We will apply Morse theory to the manifold with boundary D_R and to the function f_q . We will follow the terminology of [Dut], Section 2, p.46-47. Let us first show that f_q does not admit any inward critical point on $\Sigma_R \cap \{f_q \geq \varepsilon\}$ for R sufficiently big and ε sufficiently small (an inward critical point p is a point critical point p of $f_q|_{\Sigma_R}$ such that $\nabla f_q(p)$ is a negative multiple of $\nabla \rho(p)$). If it is not the case, then we can find a sequence of points $(x_k)_{k \in \mathbb{N}}$ in $\Gamma_{f_q, \rho}$ such that $\nabla f_q(x_k)$ is a negative multiple of $\nabla \rho(x_k)$. Using the version at infinity of the Curve Selection Lemma (see [NZ], Lemma 2), we obtain that $\lim_{k \rightarrow +\infty} f_q(x_k)$ exists and belongs to $[0, +\infty[$, which contradicts the property of Proposition 6.1.

Let us fix R sufficiently big and ε sufficiently small so that $\Sigma^*(f_q) \subset D_R$, f_q does not have inward critical points in $\Sigma_R \cap \{f_q \geq \varepsilon\}$ and

$$\chi(\{f_q \geq \varepsilon\}) = \chi(\{f_q \geq \varepsilon\} \cap D_R) \text{ and } \chi(\{f_q = \varepsilon\}) = \chi(\{f_q = \varepsilon\} \cap D_R).$$

Since f_q does not have inward critical points in $\Sigma_R \cap \{f_q \geq \varepsilon\}$, Morse theory for manifolds with boundary implies that

$$\chi(\{f_q \geq \varepsilon\} \cap D_R) - \chi(\{f_q = \varepsilon\} \cap D_R) = \sum_{i=1}^{m_q} \deg_{K_i^q} \nabla f_q. \quad (2)$$

The final result is just a combination of equality (1) and equality (2). \square

Let $F_q : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the mapping defined by $F_q(x) = q \cdot f \cdot \nabla \rho + (1 + \rho) \cdot \nabla f$. Note that for every $x \in \mathbb{R}^n$, $\nabla f_q(x) = (1 + \rho(x))^{q-1} \cdot F_q(x)$. Hence ∇f_q and F_q admit the same zeros in \mathbb{R}^n .

Corollary 6.4. *The Euler-Poincaré characteristic of X is related to F_q by the following formula :*

$$\chi(X) = 1 - \sum_{i=1}^{m_q} \deg_{K_i^q} F_q.$$

Proof. It is enough to prove that for every $i \in \{1, \dots, m_q\}$, $\deg_{K_i^q} F_q = \deg_{K_i^q} \nabla f_q$. Let us choose a relatively compact neighborhood U_i of K_i^q such that ∂U_i is a smooth manifold, $F_q^{-1}(0) \cap U_i = K_i^q = \nabla f_q^{-1}(0) \cap U_i$. The result is clear since on ∂U_i , we have $\frac{\nabla f_q}{\|\nabla f_q\|} = \frac{F_q}{\|F_q\|}$. \square

Corollary 6.5. *Let $G_q : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ be the mapping defined by $G_q(\lambda; x) = (f(x)\lambda - 1, F_q(x))$. The set $G_q^{-1}(0)$ is compact and if $R > 0$ is such that $G_q^{-1}(0) \subsetneq B_R^{n+1}$, then*

$$\chi(X) = 1 - \deg_{S_R^n} G_q.$$

Here B_R^{n+1} and S_R^n are the ball and the sphere of radius R in \mathbb{R}^{n+1} .

Proof. Since $G_q(\lambda; x) = 0$ if and only if $F_q(x) = 0$, $f(x) \neq 0$ and $\lambda = \frac{1}{f(x)}$, it is straightforward to see that $G_q^{-1}(0)$ is compact. The rest of the proof is easy. \square

These formulas are global versions of a result due to Khimshiasvili [Khi] on the Euler characteristic of the real Milnor fibre. It states that, if $g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ is an analytic function-germ with an isolated critical point at the origin, then

$$\chi(g^{-1}(\delta) \cap B_\varepsilon^n) = 1 - \text{sign } (-\delta)^n \deg_0 \nabla g,$$

for any regular value δ of g , $0 < |\delta| \ll \varepsilon \ll 1$. Here $\deg_0 \nabla g$ is the topological degree of $\frac{\nabla g}{\|\nabla g\|} : S_\varepsilon^{n-1} \rightarrow S^{n-1}$.

In their fundamental paper [OP], Petrovskii and Oleinik estimated the Euler characteristic of some real projective algebraic sets. More precisely they gave an upper bound for the following quantities :

- $|\chi(Y) - 1|$ where Y is a real projective hypersurface of even dimension,
- $|2\chi(Z_-) - 1|$ where Z_- is the subset of $\mathbb{R}P^n$ that is bounded by a real projective hypersurface Y of odd dimension and even degree and corresponds to the negative values of the polynomial that determines Y .

These results were generalized by Kharlamov [Kha1, Kha2]. In [Ar], Arnol'd found a new proof based, on Khimshiasvili's formula, and an equivalent formulation of the original Petrovskii-Oleinik inequalities. Let us state Arnol'd's version of these inequalities. We need some notations. With every n -tuples of positive integers $\mathbf{m} = (m_1, \dots, m_n)$ and with every positive integer m_0 , we will associate the following objects :

- $\Delta_n(\mathbf{m})$ is the parallelepiped in \mathbb{R}^n defined by the inequalities

$$0 \leq x_1 \leq m_1 - 1, \dots, 0 \leq x_n \leq m_n - 1,$$

- $\mu = m_1 \cdots m_n$ is the number of integral points in the parallelepiped $\Delta_n(\mathbf{m})$,
- $\nu = \frac{1}{2}(m_1 + \dots + m_n - n)$ is the mean value of the sum of the coordinates of the points in $\Delta_n(\mathbf{m})$,
- $\Pi_n(\mathbf{m})$ is the number of integral points on the central section $x_1 + \dots + x_n = \nu$ of the parallelepiped $\Delta_n(\mathbf{m})$,
- $\Pi_n(\mathbf{m}, m_0)$ is the number of integral points in $\Delta_n(\mathbf{m})$ that lie in the strip

$$\nu - \frac{1}{2}m_0 \leq x_1 + \dots + x_n \leq \nu + \frac{1}{2}m_0,$$

- $O_n(\mathbf{m}, m_0)$ is the number of integral points in $\Delta_n(\mathbf{m})$ that satisfy the inequalities

$$\nu - \frac{1}{2}m_0 \leq x_1 + \cdots + x_n \leq \nu.$$

Arnol'd [Ar] proved the following theorem.

Theorem 6.6. *Let f be a homogeneous polynomial of degree d in \mathbb{R}^n defining a non-singular hypersurface Y in $\mathbb{R}P^{n-1}$. If n is even, we have :*

$$|1 - \chi(Y)| \leq \Pi_n(\mathbf{d} - \mathbf{1}), \text{ where } \mathbf{d} - \mathbf{1} = (d - 1, \dots, d - 1) \text{ in } \mathbb{N}^n.$$

If n is odd and d is even, let Z_- be the subset of $\mathbb{R}P^n$ that is bounded by Y and corresponds to the negative values of the polynomial f . We have :

$$|1 - 2\chi(Z_-)| \leq \Pi_n(\mathbf{d} - \mathbf{1}).$$

In [Kho1] (see also [Kho2]), Khovanskii gave an affine version of this theorem.

Proposition 6.7. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial of degree d such that the surface $\{f = 0\}$ is nonsingular and the domains $\{f \leq c\}$ are compact for every $c \in \mathbb{R}$. Then the Euler-Poincaré of the domain $\{f \leq 0\}$ satisfies the inequality*

$$|1 - 2\chi(\{f \leq 0\})| \leq \Pi_n(\mathbf{d} - \mathbf{1}, d - 1),$$

where $\mathbf{d} - \mathbf{1} = (d - 1, \dots, d - 1)$ in \mathbb{N}^n .

Our aim is to give a Petroskii-Oleinik inequality for the Euler-Poincaré characteristic of any algebraic set in \mathbb{R}^n . Let X be an algebraic set in \mathbb{R}^n defined as the zero set of the polynomials f_1, \dots, f_k , each f_i having degree d_i . Hence $X = \{x \in \mathbb{R}^n \mid f(x) = 0\}$ where $f = f_1^2 + \cdots + f_k^2$. The degree of the polynomial f is $d = 2\text{Max}\{d_1, \dots, d_k\}$. The following proposition gives an upper bound for $|1 - \chi(X)|$ in terms of d .

Proposition 6.8. *Let X be an algebraic set in \mathbb{R}^n defined as the set of zeros of a nonnegative polynomial f of even degree d . We have :*

$$|1 - \chi(X)| \leq O_{n+1}(\mathbf{d} + \mathbf{1}, 2),$$

where $\mathbf{d} + \mathbf{1} = (d + 1, \dots, d + 1)$ in \mathbb{N}^{n+1} .

Proof. Let $\omega : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by $\omega(x) = x_1^2 + \cdots + x_n^2$. Applying the argument described above to the functions f and ω , we find that there exists an integer q sufficiently big and a real $R > 0$ sufficiently big such that:

$$\chi(X) = 1 - \deg_{S_R^n} G_q.$$

Let δ be a small positive regular value of G_q and let $\{p_1, \dots, p_l\}$ be the set of preimages of δ by G_q lying in S_R^n . We have :

$$1 - \chi(X) = \deg_{S_R^n}(G_q - \delta) = \sum_{j=1}^l \deg_{p_j}(G_q - \delta).$$

Since each component of $G_q - \delta$ has a degree not exceeding $d+1$, the square of the euclidian distance function in \mathbb{R}^{n+1} has degree 2 and $2 + (n+1)(d+1) \equiv n+1 \pmod{2}$, Theorem 2 of [Kho1] applied to the vector field $G_q - \delta$ and the function $R - (x_1^2 + \cdots + x_n^2 + \lambda^2)$ gives :

$$\left| \sum_{j=1}^l \deg_{p_j}(G_q - \delta) \right| \leq O_{n+1}(\mathbf{d} + \mathbf{1}, 2),$$

where $\mathbf{d} + \mathbf{1} = (d+1, \dots, d+1)$ in \mathbb{N}^{n+1} . □

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